J. Appl. Maths Mechs Vol. 55, No. 6, pp. 927–935, 1991
Printed in Great Britain.

0021–8928/91 $15.00+.00
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HYPERBOLIC VARIATIONAL INEQUALITIES IN PROBLEMS OF THE DYNAMICS OF ELASTOPLASTIC BODIES†

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(*Received* 23 *May* 1990)

Results of a study of variational inequalities appearing in dynamic problems of the theory of elastic–ideally plastic Prandtl–Reuss flow are given. The concept of a generalized solution is formulated for the general-type inequality and is used to construct the complete system of relations for a strong discontinuity. *A priori* estimates are obtained which make it possible to prove the uniqueness and continuous dependence "in the small" on time of the solutions of the Cauchy problem and initial-boundary value problems with dissipative boundary conditions, as well as the estimates of the nearness of the solutions of the variational inequality and of the system of equations with a small parameter describing the elasto-viscoplastic deformation of the bodies. The problem of the propagation of plane waves in an elastoplastic half-space with initial stresses is used as an example to illustrate the difference between the discontinuous solutions with the Mises yield condition and with the Tresca–St Venant consition in the theory of flows.

1. HYPERBOLIC VARIATIONAL INEQUALITIES

THE MODEL of an elastoplastic Prandtl–Reuss body consists, in the geometrically linear approximation, of the equations of motion, Hooke's Law, and defining relations of irreversible strain, which can be written in the form of the principle of the maximum rate of energy dissipation [1]:

$$\rho v_{i,t} = \sigma_{ij,j}, \quad e_{ij}^0 = a_{ijkl}\sigma_{kl,t} \tag{1.1}$$

$$(\sigma_{ij}^* - \sigma_{ij})\, e_{ij}^p \leqslant 0$$

$${}^1/_2\,(v_{i,j} + v_{j,i}) = e_{ij}^0 + e_{ij}^p \tag{1.2}$$

Here ρ is the density, v_i is the velocity vector in a Cartesian system of coordinates x_1, x_2, x_3, a_{ijkl} is the tensor of the elastic compliance moduli possessing the properties of symmetry and positive definiteness and e_{ij}^0, e_{ij}^p are the elastic and plastic components of the strain rate tensor.

The maximum (1.2) holds for any variation of the stress tensor σ_{ij} subjected to the following constraint:

$$f(\sigma_{ij}^*) \leqslant 1 \tag{1.3}$$

where f is the convex yield function of the material.

The system of equations (1.1), (1.2) is equivalent to the inequality

† *Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 1041–1048, 1991.

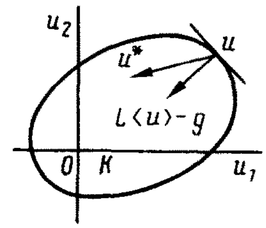

FIG. 1.

$$(v_i^* - v_i)(\rho v_{i,t} - \sigma_{ij,j}) + (\sigma_{ij}^* - \sigma_{ij})(a_{ijkl}\sigma_{kl,t} - v_{i,j}) \geqslant 0 \qquad (1.4)$$

in which the variation of the velocity vector is arbitrary.

Certain particular models of dynamic strain based on the theory of flows, for example, the models of elastoplastic plates and Timoshenko-type shells, admit of a similar formulation. A variational inequality of the most general type can be written in matrix form as follows:

$$(u^* - u)(L\langle u\rangle - g) \geqslant 0, \quad u, \ u^* \in K \qquad (1.5)$$

$$L\langle u\rangle = Au_{,t} - \sum_{s=1}^{n} B^s u_{,s} - Qu$$

Here $g = g(t, x)$ is an m-dimensional vector function defined in the domain of the solution of the problem $G \subset R^{n+1}(t, x)$, $A = A(t, x)$ and $B^s = B^s(t, x)$ are symmetrical $(m \times m)$ matrices and the matrix A is strictly positive definite, $Q = Q(t, x)$ is a $(m \times m)$ matrix, $K = K(t, x)$ is a closed convex set of admissible states and u^* is any element of K. The differential operator L is Friedrichs-hyperbolic [2].

In the case of the Prandtl–Reuss model, the vector of unknown functions $u = u(t, x)$ consists of velocities and stresses and the symmetric matrices A and B^s contain the coefficients appearing in the inequality (1.4) next to the derivatives with respect to time and the coordinates. The vector g and matrix Q are both equal to zero, and the set K is determined by the constraint (1.3).

Figure 1 shows the geometrical interpretation of the variational inequality in the space $R^m(u)$. By virtue of inequality (1.5), the angle between the vector $L\langle u\rangle - g$ and any admissible variation of the solution is acute, therefore

$$L\langle u\rangle - g = 0 \qquad (1.6)$$

if u is an internal point of the set K, and the vector $L\langle u\rangle - g$ is directed along the inner normal to the boundary of K if u lies on the boundary. In the case of a smooth boundary inequality (1.5) will be equivalent to the system of equations

$$L\langle u\rangle = g - \gamma\partial f/\partial u, \quad f \leqslant 1 \qquad (1.7)$$

in which $f(t, x, u) = \min\{r > 0\colon u/r \in K\}$ is a convex function positively homogeneous in u of the Minkovsky set K, $\gamma = \gamma(t, x)$ is a non-negative multiplier which is equal to zero when $f(u) < 1$. When $f(u) = 1$, the multiplier γ can be found from (1.7), taking into account Euler's theorem for homogeneous functions

$$\gamma = u(g - L\langle u\rangle) \qquad (1.8)$$

For the model (1.1)–(1.3), the system of quasilinear equations (1.7), (1.8) represents the formulation of relations of dynamic strain, based on the associated flow law.

2. GENERALIZED SOLUTIONS

When the matrices representing the coefficients of the operator L are sufficiently smooth, the following inequality holds for any continuously differentiable solution of the variational inequality (1.5):

$$u^*L\langle u\rangle \geqslant \frac{1}{2}(uAu)_{,t} - \frac{1}{2}\sum_{s=1}^{n}(uB^su)_{,s} - uRu + (u^* - u)g \tag{2.1}$$

$$2R = 2Q^0 + A_{,t} - \sum_{s=1}^{n} B^s_{,s},\quad 2Q^0 = Q + Q'$$

where R is a symmetric matrix, and a prime denotes the transposition operation. Multiplying both sides of (2.1) by the function $\chi = \chi(t, x) \in C^\infty(g)$, non-negative and finite in G, and integrating over the area G using Green's formula, we obtain

$$\iint_G (u^* - u/2)\left\{-(\chi A)_{,t} + \sum_{s=1}^{n}(\chi B^s)_{,s} - 2\chi Q^0\right\} u\, d\omega_x\, dt \geqslant$$

$$\geqslant \iint_G \{uL\langle u^*\rangle + (u^* - u)g\}\chi d\omega_x\, dt \tag{2.2}$$

In turn, transforming the integral in inequality (2.2) in reverse order and taking into account the arbitrariness of χ, we can obtain the inequalities (2.1) and (1.5).

The integral inequality (2.2) determines, in a natural manner, the class of generalized solutions of (1.5) containing non-smooth functions. In particular, the class contains all limits of the sequences of classical solutions converging on the norm of the space $L(^2G)$. This assertion can be proved by passage to the limit in inequality (2.2) written out for the elements of a single sequence.

Further, let $u = u(t, x)$ be a generalized solution with a strong discontinuity, satisfying inequality (2.2) for all possible admissible $u^* \in C^1(G)$, and continuously differentiable in the region G except at the hypersurface S_0 on which it has a first-order discontinuity. The hypersurface S_0 divides G into two subregions, G^+ and G^-. Transforming the integrals appearing in (2.2) over each of these subregions, we can obtain

$$\int_{S_0} [(u^* - u)Du]\chi\, d\sigma/\sqrt{1 + c^2} + \iint_G (u^* - u)(L\langle u\rangle - g)\chi\, d\omega_x\, dt \geqslant 0 \tag{2.3}$$

$$D(t, x, c, \nu) = cA + \sum_{s=1}^{n} \nu_s B^s$$

Here the square brackets denote a jump in the value of the function at the discontinuity, ν_s is the vector normal to the front of the strong discontinuity, the latter representing the intersection of S_0 by the hyperplane t = const, $c \geqslant 0$ is the velocity of propagation of the front in the direction of the normal. The unique $(n+1)$-dimensional vector $(-c, \nu)/\sqrt{1+c^2}$ is the outer normal to S_0 with respect to G^+.

The inequality (2.3) yields a condition which holds at the points of S_0, and which can be represented, by virtue of the symmetry of the matrix D, as follows:

$$(u^* - u^0) D[u] \geqslant 0, \quad u^*, \quad u^\pm \in K, \quad u^0 = (u^+ + u^-)/2 \tag{2.4}$$

where $u^\pm$ are one-sided limits of the solution on S_0.

Condition (2.4) has a geometrical interpretation analogous to that of the variational inequality (1.5). If the middle of the segment $[u^+, u^-] \subset K$ lies strictly within K, then $D[u] = 0$. If on the other hand u^0 is a boundary point, then by virtue of the convex character K, it is only when the whole segment lies on the boundary that the direction of the vector $D[u]$ coincides with the direction of its inner normal. When the boundary of K is smooth, we have the following relation:

$$D[u] = -\gamma_0 \partial f(u^0) / \partial u \tag{2.5}$$

$$\gamma_0 = \begin{cases} 0, & f(u^0) < 1 \\ -[uDu]/2 \geqslant 0, & f(u^0) = 1 \end{cases}$$

The problem of constructing discontinuous solutions in the theory of elastoplastic Prandtl–Reuss flow was first studied by Mandel in [3], who produced an erroneous argument concerning the non-uniqueness of the description of the surfaces of the velocity and stress discontinuities within the framework of this theory. A complete system of equations of strong discontinuity was given for the models of elasto–ideally plastic strain and of the strain with linear isotropic and translational hardening in [4], using the concept of maximum energy dissipation during passage across the discontinuity as the basis of the derivation. The matrix form of this system for elasto–ideally plastic media is identical with (2.5).

The impossibility of generalizing the quasilinear Prandtl–Reuss equations in the form of a complete system of integral conservation laws was shown in [5]. Thus the attempt to obtain relations at the discontinuity (1.7), (1.8) analogous to the models of ideal media [6] was unsuccessful. The formulation (1.5) yields a solution of the problem without introducing any additional concepts.

3. *A PRIORI* ESTIMATES OF THE SOLUTIONS

Let u and u' be two, sufficiently smooth solutions of the variational inequality (1.5), corresponding to various continuous right-hand sides of g and g'. Putting $u^* = u'$ in (1.5) and $u^* = u$ in the inequality written for the solution u', we can establish that

$$(u' - u)(L\langle u' - u\rangle - g' + g) \geqslant 0 \tag{3.1}$$

To obtain *a priori* estimates analogous to the estimates of the systems of linear hyperbolic equations, we construct a special region of the truncated-cone type $G = \{(t, x): t_0 \leqslant t \leqslant t_1, x \in \Omega(t)\}$, whose conical part of the surface S_c: $\varphi(t, x) = 0$ (Fig. 2) satisfies the Hamilton–Jacobi condition

$$\varphi_{,t} + H(t, x, \partial\varphi/\partial x) \geqslant 0 \tag{3.2}$$

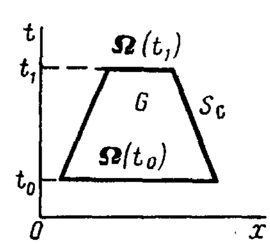

FIG. 2.

where $c = H(t, x, \nu)$ is the smallest of m real roots of the characteristic equation for the operator L: $\det D(c, \nu) = 0$. The procedure for constructing such regions is described, e.g., in [2].

Integrating inequality (3.1) over the area G we obtain

$$-\int_S (u' - u) D (u' - u)\, d\sigma/\sqrt{1 + c^2} \leqslant$$
$$\leqslant \iint_G (u' - u)\{R(u' - u) + g' - g\}\, d\omega_x\, dt \tag{3.3}$$

where S is the boundary of G and the coefficients c and ν_s appearing in the expression for the matrix D have the same meaning as before, but with reference to the hyperplane S. Condition (3.2) ensures that conditions of non-negative definiteness of the matrix $D(c, \nu)$ hold at the points S_c, therefore from (3.3), taking the Cauchy inequality into account, we obtain the expression

$$\| u' - u \|^2 (t_1) \leqslant \| u' - u \|^2 (t_0) + 2\alpha \int_{t_0}^{t_1} \| u' - u \|^2\, dt +$$
$$+ 2\beta \max_t \| g' - g \| \int_{t_0}^{t_1} \| u' - u \|\, dt, \quad \| u \|^2 = \int_{\Omega(t)} uAu\, d\omega_x \tag{3.4}$$

Here $\|u\|$ is the energetic norm, and α and β are constants depending on the elements of the matrices A and R, respectively.

This, in turn, after taking t_1, t_0 to the intermediate instant of time t, yields a differential inequality, which enables us to obtain the following estimate:

$$\| u' - u \| (t_1) \leqslant \| u' - u \| (t_0) \exp\{\alpha(t_1 - t_0)\} +$$
$$+ \beta \int_{t_0}^{t_1} \| g' - g \| (t) \exp\{\alpha(t_1 - t)\}\, dt \tag{3.5}$$

Using estimate (3.5) we can prove the uniqueness in G and the continuous dependence on the initial data and the right-hand side of the solution of the Cauchy problem: $u|_{t=0} = u_0$. It also implies the boundedness of the domain of dependence (influence) of solutions of variational inequality (1.5).

A similar estimate can be obtained in the neighbourhood of the fixed hypersurface S_u with dissipative boundary conditions specified on it. When the conditions are satisfied for the vector functions u and u', then the validity of the following inequality is ensured:

$$(u' - u) \sum_{s=1}^{n} \nu_s B^s (u' - u) \leqslant 0 \tag{3.6}$$

where ν_s is the normal to S_u, external relative to the domain of solution of the problem. In this case it is sufficient to take, as G, the part of the truncated cone separated by S_u.

The assumption that the solutions used in deriving the estimates are smooth can be weakened. Let there exist in G a hypersurface of strong discontinuities of solutions S_0. Then, by virtue of the condition at the discontinuity (2.4) taken at $u^* = (u')^0$ and analogous condition in which u and u' are interchanged, the following inequality holds:

$$[(u' - u)\, D\, (u' - u)] \leqslant 0 \tag{3.7}$$

Integrating (3.1) and using Green's formula we can obtain an inequality which differs from (3.3) in that it has an additional term, the integrand in which is identical with (3.7). From this inequality, (3.3) again follows, and then (3.5).

Let $\pi(u) = \pi(t, x, u)$ be the operator of projection onto the set K in the Euclidean metric, u be a smooth solution of variational inequality (1.5), and u' a solution of the following system of semilinear equations with a small parameter $\varepsilon > 0$:

$$L\langle u'\rangle = g - \{u' - \pi(u')\}/(2\varepsilon) \tag{3.8}$$

According to the definition of the projection operator we have

$$\{u^* - \pi(u')\}\{u' - \pi(u')\} \leqslant 0, \quad u^* \in K$$

therefore

$$(u^* - u')(L\langle u'\rangle - g) \geqslant \| u' - \pi(u')\|_0^2/(2\varepsilon) \tag{3.9}$$

Integrating the inequality obtained as a result of combining the inequality (3.9) taken at $u^* = u$, and (1.5) at $u^* = \pi(u')$, over the area of the truncated cone type, we can establish that

$$\exp(2\alpha t) d\{\| u' - u\|^2 \exp(-2\alpha t)\}/dt + \| u' - \pi(u')\|_0^2/\varepsilon \leqslant$$
$$\leqslant 2\| u' - \pi(u')\|_0 \| L\langle u\rangle - g\|_0$$

where $\|u\|_0$ is the norm of the space $L^2(G)$. After using the ε-inequality and integrating over t, this yields the following estimate:

$$\| u' - u\|^2(t_1) \leqslant \| u' - u\|^2(t_0)\exp\{2\alpha(t_1 - t_0)\} +$$
$$+ \varepsilon \int_{t_0}^{t_1} \| L\langle u\rangle - g\|_0^2(t)\exp\{2\alpha(t_1 - t)\}\, dt \tag{3.10}$$

which shows that when the initial data are the same, the norm of the difference between the solutions (1.5) and (3.8), is of the order of $\sqrt{\varepsilon}$.

The assertion of the convergence of the solutions of the system of equations (3.8) as $\varepsilon \to 0$, following from (3.10), can be generalized. If $u'(t, x, \varepsilon)$ is a family of solutions converging in $L^2(G)$ to the vector function $u(t, x)$ (not necessarily smooth), then u will be the generalized solution of the variational inequality (1.5).

Indeed, since (1.5) follows from (3.9), the integral inequality will hold for u', as well as for the limit function. Moreover, we have the identity

$$\iint_G \{\pi(u') - u'\}\psi\, d\omega_x\, dt = 2\varepsilon \iint_G (u' L'\langle\psi\rangle - g\psi)\, d\omega_x\, dt \tag{3.11}$$

where L' is an operator formally conjugated to L and $\psi = \psi(t, x)$ is a vector function finite in G and belonging to the class $C^1(G)$. Passage to the limit in this identity, taking the arbitrary form of ψ into account, yields $u = \pi(u) \in K$.

Earlier the problems of the solvability of the system of equations of elastoplastic flow, i.e. of the system (3.8) for the model (1.1)–(1.3), and the problems of convergence over the viscosity parameter ε were discussed in [1]. A generalization of the theorem of existence in the theory of elastoplastic flow, which was proved in [1] under very narrow conditions, was given in [6]. However, the concept of a generalized solution formulated in these papers does admit of strong discontinuities.

It was noted in a number of papers (for example, in [8]), that the discontinuous solution for the

Prandtl–Reuss model should naturally be regarded as the limit, with respect to the viscosity parameter, of the sequence of solutions of the system of equations of elasto-viscoplastic flow. The assertion proved here establishes the equivalence of such a concept to the concept of a generalized solution formulated in Sec. 2.

4. ELASTOPLASTIC WAVES IN A HALF-SPACE

We will construct, in closed form, the discontinuous solution of the problem of the propagation of plane stress waves produced by sudden application of a constant normal pressure $p>0$ to the surface of the half-space $x_1 \geqslant 0$, previously compressed in a transverse direction by the stress q. In the present problem the system of coordinates coincides with the principal directions of the stress tensor. The relations connecting the junps in unknown functions on the surface of the discontinuity, which are necessary for the solution, yield the variational inequality (2.4):

$$(\sigma_i^* - \sigma_i^\circ)\left\{\frac{\rho c^2}{2\mu}\left([\sigma_i] - \frac{3\lambda\sigma}{3\lambda+2\mu}\right) - [\sigma_1]\,\delta_{i1}\right\} \geqslant 0 \tag{4.1}$$

Here and henceforth $\sigma = \sigma_i\delta_{ii}/3$ is the mean (hydrostatic) stress, δ_{ij} is the Kronecker delta, and λ, μ and τ_s are the Lamé parameters of the material and yield point under pure shear.

Depending on the magnitude of the stresses p and q, the solution, according to the theory of elastoplastic flow with the Tresca–St Venant yield condition,

$$f = \max_{i,j} |\sigma_i - \sigma_j|/(2\tau_s)$$

has one, two or three surfaces of discontinuity (Fig. 3). The first one is an elastic precursor moving with the velocity of longitudinal waves $c_p = \sqrt{[(\lambda+2\mu)/\rho]}$. Before the front of the precursor we have

$$\sigma_1 = \sigma_2 = 0,\ \sigma_3 = -q\ \ (0 < q < 2\tau_s) \tag{4.2}$$

When $p \leqslant p_* = (\lambda+2\mu)\tau_s/\mu$, we have the following uniform stress state behind the front and up to the boundary of the half-space:

$$\sigma_1 = -p,\ \sigma_2 = -\lambda p/(\lambda+2\mu),\ \sigma_3 = \sigma_2 - q \tag{4.3}$$

When $p_* < p < p^* = p_* + (\lambda+2\mu)q/\mu$, a discontinuity appears corresponding to the state of incomplete plasticity, and in this case the segment $[\sigma_i^+, \sigma_i^-]$ in the stress space belongs wholly to the

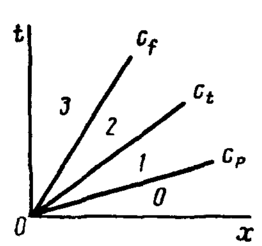

FIG. 3.

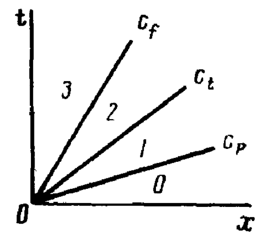

FIG. 4.

edge of a yield prism. The rate of displacement of the discontinuity is equal to $c_t = \sqrt{[(\lambda+\mu)/\rho]}$. In region 1 before the front we have

$$\sigma_1 = -p_*, \ \sigma_2 = -\lambda\tau_s/\mu, \ \sigma_3 = \sigma_2 - q \tag{4.4}$$

and behind the front and up to the boundary we have

$$\sigma_1 = -p, \ \sigma_2 = -p + 2\tau_s, \ \sigma_3 = \lambda(\tau_s - p)/(\lambda + \mu) - q \tag{4.5}$$

Finally, in the case when $p > p^*$ we have a three-wave configuration. The third discontinuity behind whose front we have the state of full plasticity, has a velocity $c_f = \sqrt{[(\lambda+2\mu/3)/\rho]}$. In region 1 the stresses are given, as before, by formulas (4.4). In region 2

$$\sigma_1 = -p^*, \ \sigma_2 = \sigma_3 = -(\lambda + \mu)q/\mu - \lambda\tau_s/\mu \tag{4.6}$$

and in region 3

$$\sigma_1 = -p, \ \sigma_2 = \sigma_3 = -p + 2\tau_s \tag{4.7}$$

According to the assertion proved above, the solution of the problem given here is unique in the class of piecewise discontinuous functions. The unique solution of this problem in the theory of flows with Mises condition

$$f = \sqrt{\sum_{i,j}(\sigma_i - \sigma_j)^2}/(2\sigma_s)$$

where σ_s is the tensile yield point, has a single discontinuity, namely the elastic precursor, irrespective of the value of p. When $p \leqslant p_*' = (\lambda+2\mu)(q+\sqrt{4\sigma_s^2 - 3q^2})/(4\mu)$, the stress state in the half-space is given by formulas (4.2) and (4.3). For large p we have, in region 1 (Fig. 4) behind the front of the precursor,

$$\sigma_1 = -p_*', \ \sigma_2 = -\lambda p_*'/(\lambda + 2\mu), \ \sigma_3 = \sigma_2 - q \tag{4.8}$$

A self-similar solution depending on the variable $c = x/t$ is adjacent continuously to state 1, and the following system of ordinary differential equations can be obtained from (1.7), (1.8) for determining this solution:

$$(\rho c^2 - \lambda - 2\mu)d\sigma_1/dc = \gamma'(\sigma_1 - \sigma)$$
$$\rho c^2 d\sigma_i/dc - \lambda d\sigma_1/dc = \gamma'(\sigma_i - \sigma), \ (i = 2, 3)$$
$$\gamma' = -3\mu\sigma_s^{-2}(\sigma_1 - \sigma)\, d\sigma_1/dc \geqslant 0$$

The surface of the join of these solutions propagates at the rate of $c_* = \sqrt{c_f^2 + \mu q^2/(\rho\sigma_s^2)}$. In region 2 the expression for the mean stress has the form

$$3\sigma = \frac{3\lambda + 2\mu}{4\mu}\sigma_s \ln\frac{\varphi(c)}{\varphi(c_*)} - \frac{3\lambda + 2\mu}{\lambda + 2\mu}p_*' - q \tag{4.9}$$

and the principal stresses are:

$$\sigma_1 = \sigma - 2\sigma_s\varphi_p(c)/3, \quad \sigma_2 = \sigma + \sigma_s\{\varphi_p(c)/3 + \varphi_f(c)/\sqrt{3}\},$$
$$\sigma_3 = \sigma + \sigma_s\{\varphi_p(c)/3 - \varphi_f(c)/\sqrt{3}\} \tag{4.10}$$

Here

$$\varphi = \frac{1+\varphi_p(c)}{1-\varphi_p(c)}, \quad \varphi_p = \left(\frac{c_p^2-c^2}{c_p^2-c_f^2}\right)^{1/2}, \quad \varphi_f = \left(\frac{c^2-c_f^2}{c_p^2-c_f^2}\right)^{1/2}$$

The rate of motion of the boundary of the region $2c = c^*$ is a root of the equation

$$\sigma_1(c) + p = 0 \tag{4.11}$$

Such a root exists when $p > p_*$ in the interval (c_*, c_f), since the function on the left-hand side of (4.11) changes its sign. In region 3 we have a homogeneous stress state determined by formulas (4.9) and (4.10) when $c = c^*$.

In the limit, as $q \to 0$, the relation $c_* = c^* = c_f$ holds. Thus, in the case of $q = 0$, a second surface of strong discontinuity appears, moving at a rate of c_f. The corresponding stresses are piecewise-constant functions.

The solutions given here can be used as tests when constructing numerical methods.

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Translated by L.K.